EE 538

FALL 2000

HOMEWORK #1 Solutions

1.6 (a) Assume $x_a(t) = A \cdot \cos(2\pi F_0 t + \theta)$. Then

$$x(n) = A \cdot \cos(2\pi F_0 n T + \theta)$$
$$= A \cos\left(2\pi n \frac{T}{T_p} + \theta\right)$$

For periodicity, we require $x(n) = x(n+N)$ where N is the fundamental period of $x(n)$.
(N: integer)

i.e. $A \cos\left(2\pi n \frac{T}{T_p} + \theta\right) = A \cos\left(2\pi (n+N) \frac{T}{T_p} + \theta\right)$

This is true iff there exists an integer k such that

$$2\pi N \frac{T}{T_p} = 2\pi k \qquad \text{i.e.} \quad \boxed{\frac{T}{T_p} = \frac{k}{N}}$$

(b) Fundamental period of $x(n)$ in seconds
$\rightarrow N \cdot T$

(c) From (a). $N \cdot T = k \cdot T_p$.
i.e. fundamental period of $x(n)$, in seconds, is equal to an integer number of periods of $x_a(t)$.

1.7. (a) ≥ 20 kHz
(b) Aliasing. Generating frequency $F_1' = -3$ KHz
(c) Aliasing. Generating frequency $F_2' = 1$ kHz

1.8. (a) 200 Hz
(b) 125 Hz

1.9 (a) $F_1 = 240$ Hz $\quad F_2 = 360$ Hz

$$\therefore F_N = 2 \cdot F_{max} = 720 \text{ Hz}$$

(b) $F_s = 600$ Hz. Folding frequency $= \frac{F_s}{2} = 300$ Hz.

(c). $x(n) = x_a(nT) = x_a\left(\frac{n}{F_s}\right) = \sin 2\pi \cdot \left(\frac{2}{5}\right) n + 3 \sin 2\pi \cdot \left(\frac{3}{5}\right) n$

$$= \sin 2\pi \cdot \left(\frac{2}{5}\right) n + 3 \cdot \sin 2\pi \left(-\frac{2}{5}\right) n$$

$$= -2 \sin 2\pi \left(\frac{2}{5}\right) n$$

$\therefore$ Frequency of $x(n) = \frac{2}{5} \cdot 2\pi = \frac{4\pi}{5}$ (radians/sample)

(d) $y_a(t) = -2 \sin 2\pi \cdot \frac{2}{5} \cdot 600t = -2 \sin 480\pi t$.

1.11 $x_a(t) = 3 \cos 100\pi t + 2 \sin 250\pi t$

$F_s = 1/T = 200$ samples/s.

$\therefore x(n) = 3 \cos\left(\frac{100\pi}{200} n\right) + 2 \sin\left(\frac{250\pi}{200} n\right) = 3 \cos 2\pi\left(\frac{1}{4}\right) n + 2 \sin 2\pi\left(\frac{5}{8}\right) n$

$$= 3 \cos 2\pi \cdot \left(\frac{1}{4}\right) n - 2 \sin 2\pi \left(\frac{3}{8}\right) n$$

$F_s' = 1/T' = 1000$ samples/s.

$\therefore y_a(t) = 3 \cos 2\pi \cdot \left(\frac{1}{4}\right) \cdot 1000t - 2 \sin 2\pi \cdot \left(\frac{3}{8}\right) \cdot 1000t$

$$= 3 \cos 500\pi t - 2 \sin 750\pi t.$$

2.10 H ~ time invariant.

$$x_1(n) = \{\underset{\uparrow}{1}, 0, 2\} \xrightarrow{H} y_1(n) = \{\underset{\uparrow}{0}, 1, 2\}$$

$$x_2(n) = \{\underset{\uparrow}{0}, 0, 3\} \xrightarrow{H} y_2(n) = \{\underset{\uparrow}{0}, 1, 2\}$$

$$x_3(n) = \{\underset{\uparrow}{0}, 0, 0, 1\} \longrightarrow y_3(n) = \{1, \underset{\uparrow}{2}, 1\}$$

Is H linear ?

$$h(n) = H[\delta(n)] = H[x_3(n+3)] = y_3(n+3) = \cancel{\{1, 2, 1, 0\}} \ \{1, 2, 1, 0, \underset{\uparrow}{0}\}$$

$$x_2(n) = 3\delta(n-2) \Rightarrow y_2(n) \overset{?}{=} 3h(n-2)$$

$$3h(n-2) = 3\{1, 2, \underset{\uparrow}{1}\} = \{3, 6, \underset{\uparrow}{3}\} \neq y_2(n)$$

$\therefore$ H is not linear.

2.11 H ~ linear.

$$x_1(n) = \{-1, \underset{\uparrow}{2}, 1\} \xrightarrow{H} y_1(n) = \{1, \underset{\uparrow}{2}, -1, 0, 1\}$$

$$x_2(n) = \{1, \underset{\uparrow}{-1}, -1\} \xrightarrow{H} y_2(n) = \{-1, \underset{\uparrow}{1}, 0, 2\}$$

$$x_3(n) = \{0, \underset{\uparrow}{1}, 1\} \xrightarrow{H} y_3(n) = \{\underset{\uparrow}{1}, 2, 1\}$$

Is H time invariant ?

$$x_4(n) = x_1(n) + x_2(n) = \{0, \underset{\uparrow}{1}, 0\} \quad y_4(n) = y_1(n) + y_2(n) = \{0, \underset{\uparrow}{3}, -1, 2, 1\}$$

$$x_5(n) = x_2(n) + x_3(n) = \{1, \underset{\uparrow}{0}, 0\} \quad y_5(n) = y_2(n) + y_3(n) = \{-1, \underset{\uparrow}{2}, 2, 3\}$$

$x_4(n) = x_5(n-1)$ But $y_4(n) \neq y_5(n-1)$

$\therefore$ H is not time invariant.

2.13. Prove: A relaxed LTI system is BIBO stable iff

$$\sum_{n=-\infty}^{\infty} |h(n)| \leq M_h < \infty$$

for some constant M_h.

proof: (if part)

Assume $\sum_{k=-\infty}^{\infty} |h(k)| \leq M_h < \infty$ and prove that if $|x(n)| < M_i \ \forall n$, then

$|y(n)| < M_o \quad \forall n$.

$$|y(n)| < \left|\sum_{k=-\infty}^{\infty} x(n-k)h(k)\right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x(n-k)\, h(k)| \qquad \text{(Triangle Inequality)}$$

$$= \sum_{k=-\infty}^{\infty} |x(n-k)| \cdot |h(k)|$$

$$\leq \sum_{k=-\infty}^{\infty} M_i \cdot |h(k)| \leq M_i \cdot M_h = M_o \qquad \forall n \text{ iff } \sum_{k=-\infty}^{\infty} |h(k)| \leq M_k$$

$$\Rightarrow |y(n)| \leq M_o \quad \forall n.$$

(Only if part)

Now show that a bounded input may give rise to an unbounded output if $\sum_{h=-\infty}^{\infty} |h(n)| = \infty$.

Consider $x(n) = \begin{cases} \dfrac{h^*(-n)}{|h(-n)|} & \text{for } h(-n) \neq 0 \\ 0 & \text{for } h(-n) = 0 \end{cases}$

Note: $|x(n)| = \begin{cases} 1 & h(-n) \neq 0 \\ 0 & h(-n) = 0 \end{cases}$

Thus, this input is bounded $\forall n$.

$$\Rightarrow y(0) = \sum_{k=-\infty}^{\infty} x(-k)h(k) = \sum_{k=-\infty}^{+\infty} \frac{h^*(k)h(k)}{|h(k)|} = \sum_{k=-\infty}^{+\infty} |h(k)| = \infty$$

Thus, a bounded input gives rise to an unbounded output if $\sum_{n=-\infty}^{\infty} |h(n)| = \infty$. Therefore, for a system to be BIBO stable, it is necessary that $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

2.45. $y(n) = a y(n-1) + b x(n)$

(a) Assume system is relaxed, i.e. $h(n) = 0 \quad \forall n < 0$.

$$h(n) = y_{zs}(n) \Big|_{x(n)=\delta(n)}$$

$$\therefore h(0) = a h(-1) + b\delta(0) = b.$$

$$h(1) = a h(0) + b\delta(1) = ab.$$

$$\vdots$$

$$h(n) = a^n b\, u(n)$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} a^n b = \frac{b}{1-a} \qquad (a < 1)$$

$$\frac{b}{1-a} = 1 \Rightarrow b = 1-a.$$

(b). $S(n) = y_{zs}(n) \Big|_{x(n)=u(n)}$

$$S(0) = aS(-1) + bu(0) = b.$$

$$S(1) = aS(0) + bu(1) = ab + b$$

$$S(2) = a^2 b + ab + b.$$

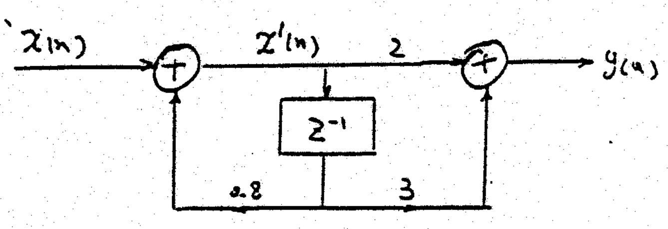
$$\vdots$$

$$S(n)=\sum_{k=0}^{n} b a^k = b\cdot\frac{1-a^{n+1}}{1-a}u(n) \qquad \left(\text{for } |a|<1\right)$$

$$S(\infty)=\frac{b}{1-a}=1 \quad\Rightarrow\quad b=1-a.$$

(c) $$S(\infty)=\sum_{n=-\infty}^{\infty} h(n) = \frac{b}{1-a}$$

2.46



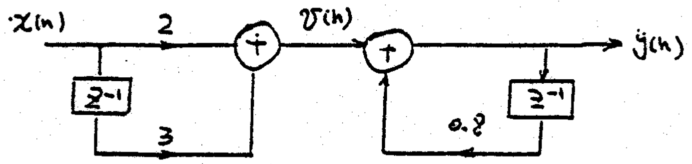
(c) $$x'(n)=x(n)+0.8x'(n-1)$$
$$y(n)=2x'(n)+3x'(n-1)$$

Let $g(n)=x'(n)\big|_{x(n)=\delta(n)}$ then

$$g(0)=\delta(0)+0.8x'(-1)=1+0=1$$
$$g(1)=0.8g(0)=0.8$$
$$\vdots$$
$$g(n)=(0.8)^n u(n).$$

$$\therefore\ h(n)=2g(n)+3g(n-1)=2(0.8)^n u(n)+3\cdot(0.8)^{n-1}u(n-1)$$
$$=2(0.8)^0\delta(n)+2(0.8)^n u(n-1)+\frac{3}{0.8}(0.8)^n u(n-1)$$
$$=2\delta(n)+5.75(0.8)^n u(n-1)$$
$$=2\delta(n)+4.6(0.8)^{n-1}u(n-1)$$

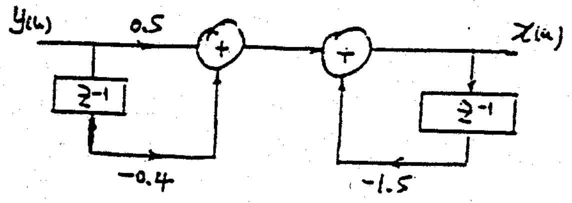
(b) The direct Form I Realization for this system is:



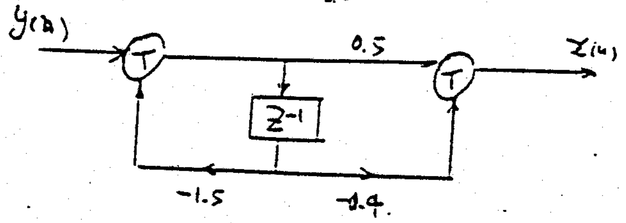
$$y(n) = v(n) + 0.8\, y(n-1) \quad \Rightarrow \quad v(n) = y(n) - 0.8\, y(n-1)$$

$$v(n) = 2x(n) + 3x(n-1) \quad \Rightarrow \quad x(n) = \tfrac{1}{2} v(n) - 1.5\, x(n-1)$$

$$= \tfrac{1}{2} y(n) - 0.4\, y(n-1) - 1.5\, x(n-1)$$



Direct Form I



Direct Form II

2.61 $x(n) = s(n) + r_1 s(n-k_1) + r_2 s(n-k_2)$

(a) $r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)$

$$= \sum_{n=-\infty}^{\infty} \left[s(n) + r_1 s(n-k_1) + r_2 s(n-k_2)\right]\left[s(n-l) + r_1 s(n-l-k_1) + r_2 s(n-l-k_2)\right]$$

$$= r_{ss}(l) + r_1 r_{ss}(l+k_1) + r_2 r_{ss}(l+k_2)$$

$$+ r_1 r_{ss}(l-k_1) + r_1^2 r_{ss}(l) + r_1 r_2 r_{ss}(l-k_1+k_2)$$

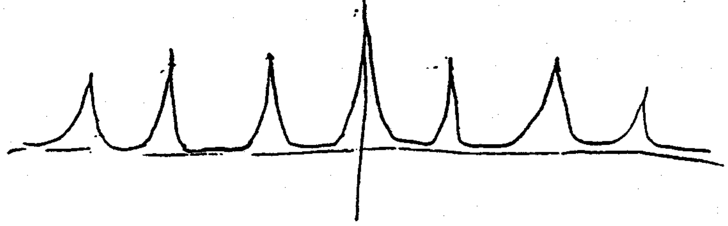
$$+ r_2 r_{ss}(l-k_2) + r_1 r_2 r_{ss}(l+k_1-k_2) + r_2^2 r_{ss}(l)$$

b) If $r_{ss}(l)$ has a sharp peak at $l=0$. $r_{ss}(l-k)$ will have a sharp peak at $l=k$.

In our case
$r_{xx}(l)$ will have peaks at $l=0$, $\pm k_1$, $\pm k_2$, $\pm|k_2-k_1|$

We have 7 symmetrically distributed peaks with the largest at $l=0$

Envelope for $|r_{xx}|$ looks like



d) If we assume that k_2 is the larger lag value of the components of $x(n)$ we can find k_2 (corresponding to outermost peaks) as

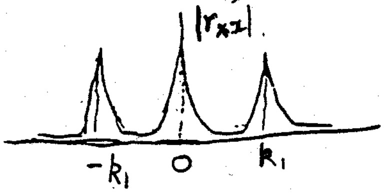
$$|k_2-k_1| < |k_2| \qquad \text{and} \quad |k_1| < |k_2|$$

Seperating and identifying the peaks corresponding to k_1 and k_2-k_1 though is impossible, as the middle two peaks between $l=0$ and $l=\pm k_2$, could correspond to either. Hence determining r_2 is possible but determining r_1 is impossible

c) If $r_2=0$ we get

$$r_{xx}(l) = (1+r_1^2)\, r_{ss}(l) + r_1 r_{ss}(l+k_1) + r_1 r_{ss}(l-k_1)$$

Here k_1 can be determined exactly



Ratio of peaks at 0 & k_1

$$\boxed{\frac{\text{Peak at } 0}{\text{Peak at } k_1} \doteq \frac{1+r_1^2}{r_1}}$$

Solve above quadratic equation and take positive value for r_1